Index Models

Chapter 10

The Need for a "Simpler" Model

- the input list (i.e., list of assets on the market) in the Markovitz portfolio selection model is very important, determining the accuracy of finding "efficient" portfolios
- however, it involves a lot of calculations
- for example, for $n = 50$ assets we need to calculate (or, more precisely, estimate):
  - $n = 50$ expected returns
  - $n = 50$ variances
  - $n(n-1) / 2 = 1225$ covariances
- in total, $n(n+3) / 2 = 1325$ estimates!

The Need for a "Simpler" Model (cont.)

- also, because we estimate returns, variances and covariances, small errors can have large effects
- for example, if we estimate wrongly the covariance matrix (mutually inconsistent correlation coefficients), it is possible that the variance of the portfolio we construct is *negative*!
- hence, the need for a simpler model, that doesn't rely on calculations that many calculations

The Single-Factor Model

- security returns tend to move together because of market risk
- suppose we can summarize all the common effects into one macroeconomic variable
- then we can write the return on stock i as

  $$r_i = E(r_i) + m_i + e_i$$

  where m_i is the unanticipated effect of the common macroeconomic factors, and e_i is the unanticipated effect of firm-specific factors
- note that $E(m_i) = E(e_i) = 0$

The Single-Factor Model (cont.)

- different firms have different sensitivities to macroeconomic events
- denote the sensitivity of firm i to the common set of factors (*sensitivity coefficient*) by β_i
- denote the variable that encompasses the unanticipated effect of the common set of macroeconomic factors by F
- then $m_i = \beta_i F$ and the equation for the return on stock k becomes the *single-factor model*:

$$r_i = E(r_i) + \beta_i F + e_i$$

The Single-Index Model

- now, we need a measure for F, the common macroeconomic factors
- since a market index corresponds to a well-diversified portfolio, its return should respond only to the common macroeconomic factors
- hence, we can use a market index (say, S&P 500) to approximate our macroeconomic variable ➔ the *single-index model*
- investors are more interested in *risk premiums* rather that *returns*

The Single-Index Model (cont.)

- then we can write the return on stock *k* as

  $$r_i - r_f = \alpha_i + \beta_i [r_m - r_f] + e_i$$

  or:

  $$R_i = \alpha_i + \beta_i R_m + e_i$$

  where R_i, R_m are excess returns
- we can decompose the excess return on a security into three components:
  - α_i = return if the excess return on the market portfolio is zero
  - $\beta_i [r_m - r_f]$ = return due to market movements
  - e_i = return due to unexpected firm-specific factors

Why Beta?

- covariance between returns on stock k and market portfolio (index):

$$
\begin{aligned}
Cov(r_i, r_m) &= Cov(r_i - r_f, r_m) = Cov(r_i - r_f, r_m - r_f) \\
&= Cov(R_i, R_m) \\
&= Cov(\alpha_i + \beta_i R_m + e_i, R_m) \\
&= Cov(\alpha_i, R_m) + Cov(\beta_i R_m, R_m) + Cov(e_i, R_m) \\
&= 0 + \beta_i Cov(R_m, R_m) + 0 \\
&= \beta_i \sigma_m^2
\end{aligned}
$$

- hence, $\beta_i = \dfrac{Cov(r_i, r_m)}{\sigma_m^2}$

Return Variance and Covariances

- note that the variance of returns is
  $$\sigma_i^2 = Var(\alpha_i + \beta_i R_m + e_i) = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2$$
- hence, there are two sources of risk:
  - *market* (*idiosyncratic*) risk: $\beta_i^2 \sigma_m{}^2$
  - *firm-specific* risk: $\sigma_{ei}{}^2$
- covariance between returns on stocks i and j:
  $$\begin{aligned} Cov(r_i, r_j) &= Cov(R_i, R_j) \\ &= Cov(\alpha_i + \beta_i R_m + e_i, \alpha_j + \beta_j R_m + e_j) \\ &= \beta_i \beta_j \sigma_m^2 \end{aligned}$$

Why Is It a Simpler Model?

- the input list for $n = 50$ assets consists of:
  - $n = 50$ estimates of expected returns
  - $n = 50$ estimates of sensitivity coefficients (β_i)
  - 1 (one) estimate of the variance of the market portfolio (index)
  - $n = 50$ estimates of firm-specific risks (σ_{ei}^2)
- in total, $3n + 1 = 151$ estimates, as compared to 1325!

Advantages and Disadvantages

- Advantages:
  - easier to generate the Markovitz frontier
  - allows the specialization of security analysts by industry
- Disadvantages:
  - overly simplistic decomposition of risk – macro vs. micro (ignores, for example, industry specific events)
  - resulting portfolios might be inefficient

Estimating the Index Model

- the single index model equation has the form of a *regression equation*:

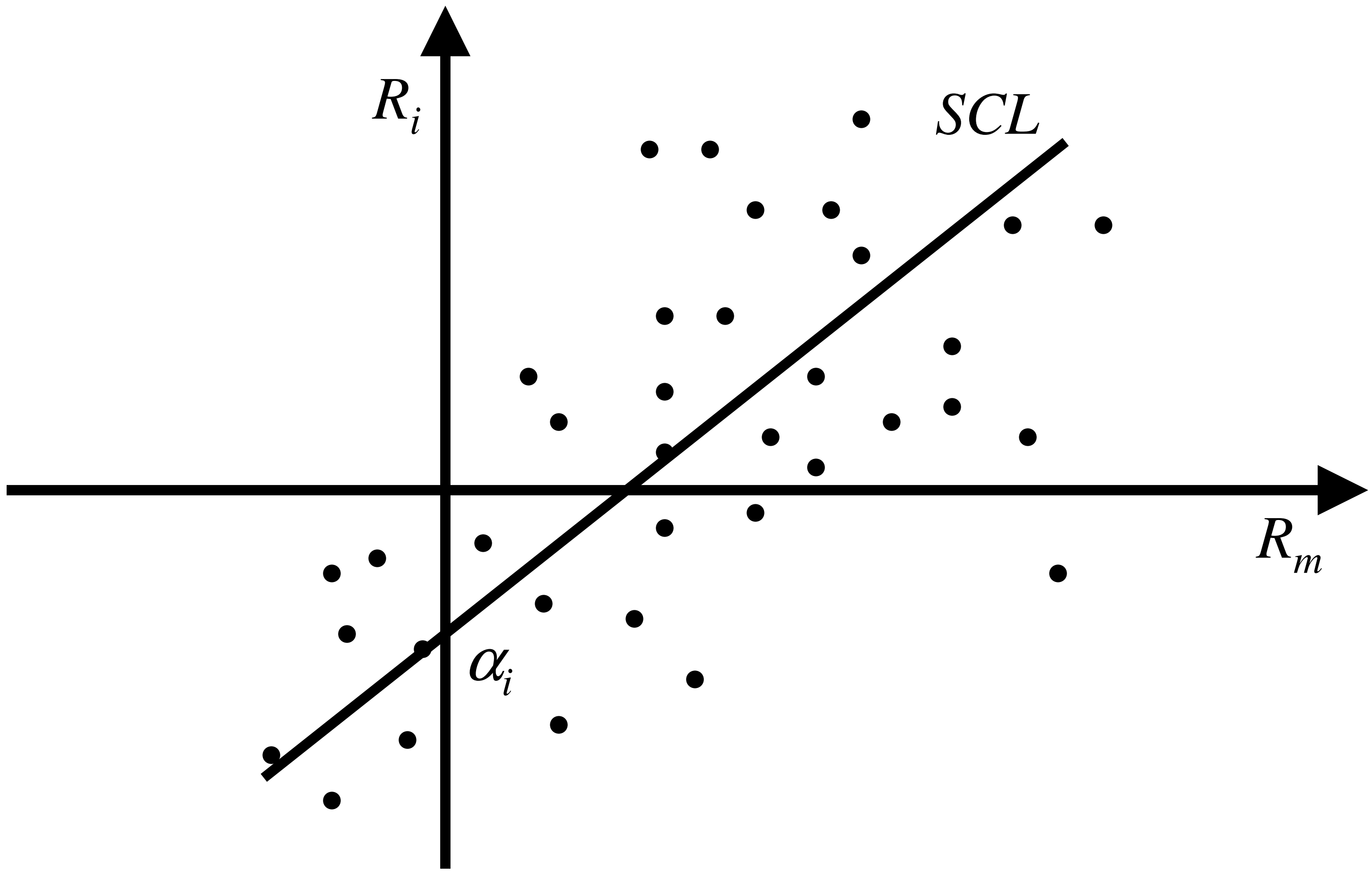
$$R_{it} = \alpha_i + \beta_i R_{mt} + e_{it}$$

- this defines a line with intercept α_i and slope β_i, with e_{it} being the deviations from the line for the individual returns
- this line is called the *Security Characteristic Line* (*SCL*)
- it can be estimated using standard estimation techniques

Estimating the Index Model (cont.)

- to do that, follow the steps:
  - gather historical data on stock prices (usually closing price), market index and risk-free asset (T-bills)
  - construct one-period returns (for a one-month or one-week holding periods) for the stock, the market index and the risk-free asset
  - this yields the variance of the return on the market index
  - construct excess returns for the stock and the market index
  - estimate the index model equation and obtain estimates of α_i, β_i, and σ_{ei}^2

Security Characteristic Line



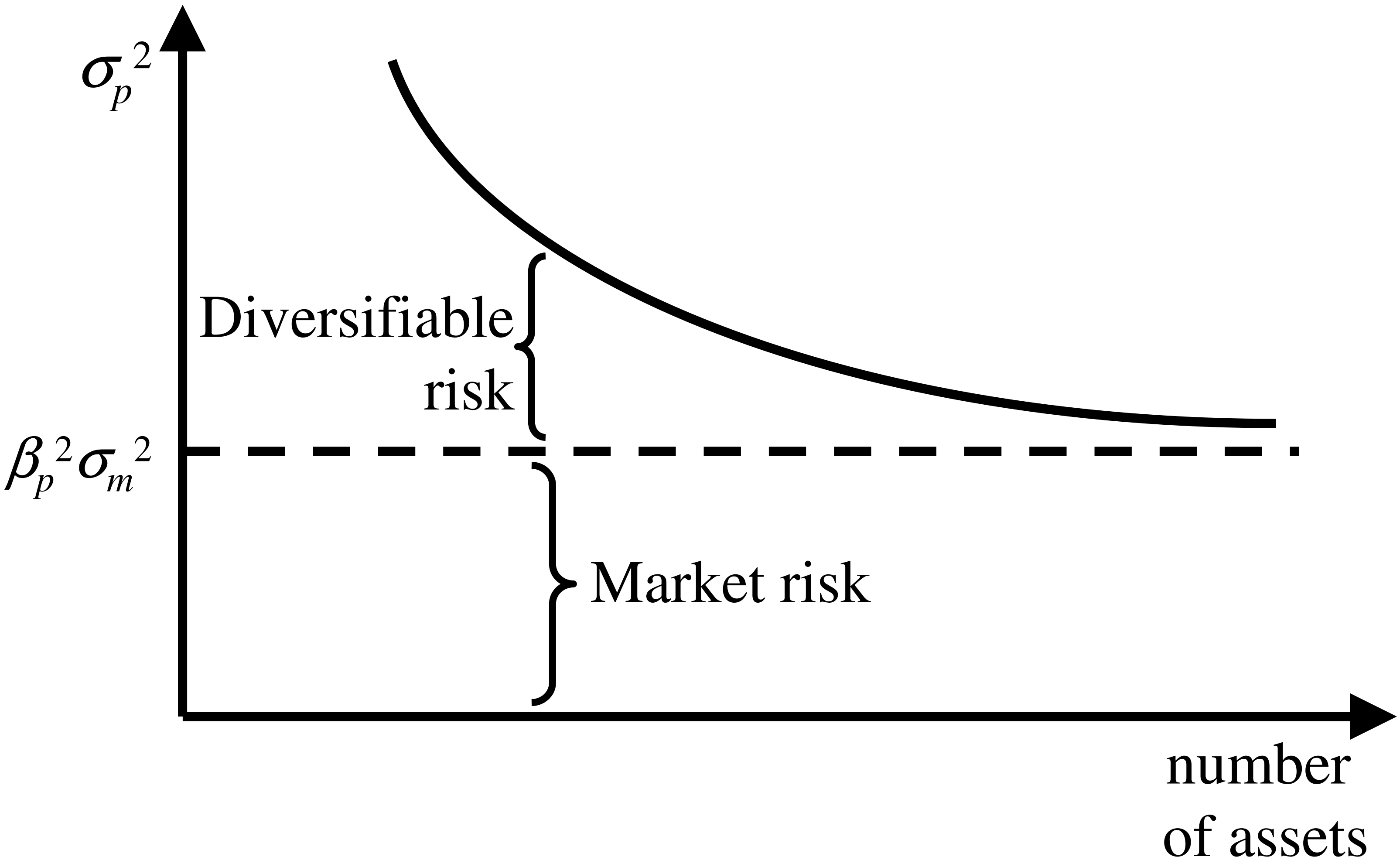
Portfolio Risk

- the single index model equation for a portfolio has the same form:

  $$R_p = \alpha_p + \beta_p R_m + e_p$$

  where α_p, β_p, and e_p are weighted returns of the individual stock counterparts
- the variance of the "firm-specific" term e_p decreases as the number of stocks included in the portfolio increases
- this is another example of the effects of diversification on risk

The Effect of Diversification on Risk



Problems with the CAPM

- remember that the CAPM holds that

$$E(r_i) = r_f + \beta_i \, [E(r_m) - r_f]$$

- implication of the CAPM:
  - the market portfolio is efficient
  - relationship between risk and *expected* returns
- in practice, the CAPM is *not* directly testable, because it makes prediction about *ex ante* returns, while we only observe *ex post* returns

Testing the CAPM using the Index Model

- remember that the beta coefficient in the index model is the same as the beta in the CAPM
- we can write the index model equation as

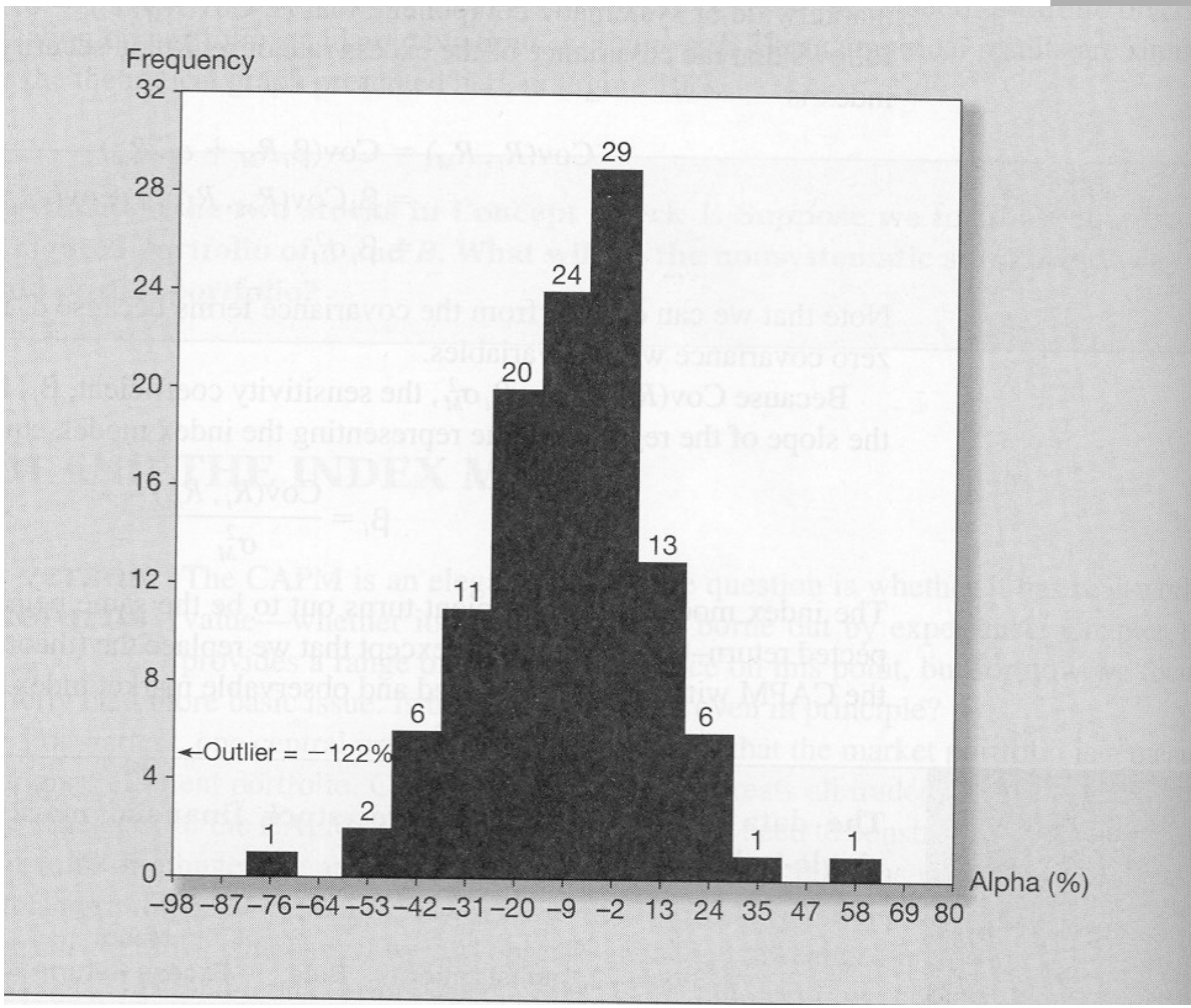
$$r_i - r_f = \alpha_i + \beta_i [r_m - r_f] + e_i$$

- take expectations of both sides:

$$E(r_i) - r_f = \alpha_i + \beta_i [E(r_m) - r_f]$$

- according to CAPM, a stock's α should be equal to zero, on average
- hence, we should find that our estimates of α's are centered around zero (Jensen, 1968)

Estimated Alphas



More Practical Insights

- the beta coefficient, variances of return on market index and of firm-specific deviations can be estimated from historical data
- a source of such information is Merrill Lynch's *Security Risk Evaluation* book (*beta book*)
- differences from index model:
  - uses returns rather than excess returns
  - ignores dividends

Adjusted Beta

- estimated beta coefficients tend to move toward one over time
- reasons:
  - average beta for all stocks is 1 (market beta)
  - firms become more diversified over time ➔ they eliminate more of firm-specific risk
- Merrill Lynch calculates an *adjusted beta* to compensate for this tendency:

$$\beta^a = \frac{2}{3}\beta^e + \frac{1}{3}$$

where β^a and β^e are adjusted and estimated betas, respectively

Tracking Portfolios

- suppose an investor identifies an underpriced portfolio P ($\alpha_p > 0$) and wants to invest in it
- still, if the market as a whole declines, she would still end up losing money
- to avoid that, she can construct a *tracking portfolio T*, with the following structure:
  - a proportion β_p in the market index
  - a proportion $(1 - \beta_p)$ in the risk-free asset
- since T is constructed from the market index and the risk-free asset, its alpha coefficient is zero

Tracking Portfolios (cont.)

- next, she can buy *P* and short-sell *T* at the same time ➔ eliminates the market risk
- still, the investment will yield a return (because of the portfolio *P*'s positive alpha)
- note: the portfolio is *not* risk-free – it still has the firm-specific risk
- this strategy is what many hedge funds do